Methods of taking inverse z transforms

	analytic (equation) eg $(-0.6)^n u[n]$	numeric (sequence of numbers) eg [6 2 9 3]
hand	PFD	long division
Matlab	residuez	filter

Partial Fraction Decomposition (PFD)

1. • Write $X(z)$ as unfactored polynomial in z^{-1}, eg $\dfrac{\overset{\text{always a constant}}{\textcircled{2}}+6z^{-1}+\dots+3z^{-\textcircled{M}}}{\underset{\text{always 1}}{\textcircled{1}}-2z^{-1}+\dots+6z^{-\textcircled{N}}}$

   • Make proper, ie $N > M$.
   If not, <u>long division in reverse order</u> (increasing powers of z)

   <u>ex</u>

$$\frac{4+5z^{-1}+28z^{-2}-15z^{-3}}{1-2z^{-1}-15z^{-2}} \qquad \left.\begin{matrix} M=3 \\ N=2 \end{matrix}\right\} \text{improper}$$

$$\begin{array}{rl} & z^{-1} - 2 \\ -15z^{-2}-2z^{-1}+1 & \overline{\big)\,-15z^{-3}+28z^{-2}+5z^{-1}+4} \\ & \underline{-15z^{-3} - 2z^{-2} + z^{-1}} \\ & \qquad\qquad 30z^{-2}+4z^{-1}+4 \\ & \qquad\qquad \underline{30z^{-2}+4z^{-1}-2} \\ & \qquad\qquad\qquad\qquad\qquad 6 \end{array}$$

← note <u>reversed order</u>

6 ← stop here because any further and you would get z raised to positive powers in the result

So as a proper fraction,

$$\boxed{z^{-1}-2+\frac{6}{1-2z^{-1}-15z^{-2}}}$$

2. Factor the denomenator and use PFD. Form: $\dfrac{\text{numerator}}{(1-\lambda_1 z^{-1})(1-\lambda_2 z^{-1})\dots}$

   where $\lambda_1, \lambda_2, \dots$ are poles

   A. Real, unique poles <u>ex</u>

$$\frac{6+z^{-1}}{1-0.8z^{-1}-0.2z^{-2}}$$

poles: $1-0.8z^{-1}-0.2z^{-2}=0$

$z^2-0.8z-0.2=0$ Use quadratic formula with $a=1, b=-0.8, c=-0.2$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = 1, -\frac{1}{5}$$

$$= \frac{6+z^{-1}}{(1-z^{-1})(1+\frac{1}{5}z^{-1})}$$

$$= \frac{A}{1-z^{-1}} + \frac{B}{1+1/5\,z^{-1}} \quad \text{where}$$

$A = \dfrac{6+1}{1+0.2} = 5.8$ (sub $z^{-1} \Rightarrow 1$, first root)

$B = \dfrac{6-5}{1-(-5)} = \dfrac{1}{6}$ (sub $z^{-1} \rightarrow -5$, second root)

$$= \frac{5.8}{1-z^{-1}} + \frac{1/6}{1+0.2z^{-1}}$$

$$\Leftrightarrow \boxed{5.8\,u[n] + 1/6\,(-0.2)^n u[n]}$$

B. Real, repeated poles ex $\frac{\text{numerator}}{(1-3z^{-1})^2(1-2z^{-1})}$

use residuez (discussed next) or hand multiplication to get in this form:

$$\begin{array}{ccccc} \frac{a}{1-3z^{-1}} & + & \frac{b}{(1-3z^{-1})^2} & + & \frac{c}{1-2z^{-1}} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \end{array}$$

$$x[n] = [a \cdot 3^n + b \cdot 3^n (n+1) + C \cdot 2^n] u[n]$$

Use the tables for this last step

$X(z)$	$x[n]$
1	$\delta[n]$
z^{-1}	$\delta[n-1]$
z	$\delta[n+1]$
$\frac{1}{1-az^{-1}}$	$a^n u[n]$
$\frac{1}{(1-az^{-1})^2}$	$a^n (n+1) u[n]$

PFD using Matlab: residuez

Ex $X(z) = \dfrac{5 - 5.2z^{-1} + 2.4z^{-2} + 0.2z^{-4}}{1 - 1.8z^{-1} + 0.6z^{-2} + 0.2z^{-3}}$ (unfactored poly in z^{-1})

$$\gg [r,p,k] = \text{residuez}(\overbrace{[5,-5.2,2.4,0,0.2]}^{\text{numerator}}, \overbrace{[1,-1.8,0.6,0.2]}^{\text{denomenator}})$$

r = [0 2 6] the results of the "cover up" method

p = [1 1 -0.2] the poles, aka roots

k = [-3 1] was improper, this is the divided-out part

$$X(z) = -3 + z^{-1} + \frac{\overset{r(1)}{0}}{\underset{p(1)}{1 - z^{-1}}} + \frac{\overset{r(2)}{2}}{\underset{p(2)}{(1 - z^{-1})^2}} + \frac{\overset{r(3)}{6}}{\underset{p(3)}{1 + 0.2z^{-1}}}$$

Inverse Z using long division

unfactored poly in z^{-1}

Ex. $X(z) = \dfrac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$ quickly find $x[0]$, $x[1]$

Could do full PFD by hand, then sub in $n=0,1$, but takes long time. Instead, long division to just find the first few samples

Do long division without reversing coefficient order, unlike PFD

$$\begin{array}{r|l} & \boxed{1 + 1.6z^{-1} + \dots} \text{ keep going for more samples} \\ \hline 1 + 0.4z^{-1} - 0.12z^{-2} & 1 + 2z^{-1} \\ & \underline{1 + 0.4z^{-1} - 0.12z^{-2}} \\ & \quad 1.6z^{-1} + 0.12z^{-2} \\ & \quad 1.6z^{-1} + \dots \end{array}$$

So $X(z) = 1 + 1.6z^{-1} + \dots$

$\Leftrightarrow x[n] = \delta[n] + 1.6\,\delta[n-1] + \dots$

$\Rightarrow \boxed{x[0] = 1, \quad x[1] = 1.6}$ Much faster than PFD!

Inverse Z samples using Matlab: filter

idea: inverse z transform of $H(z)$ is $h[n]$, the system response to an impulse so, use filter(num, den, x) with x = [1 0 0 0 ...]

Ex. Find first 100 values of $x[n]$ if $X(z) = \dfrac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$

$$x = \text{filter}(\overbrace{[1,2]}^{\text{num}}, \overbrace{[1, 0.4, -0.12]}^{\text{den}}, [1, \text{zeros}(1, 99)])$$