DFT - DTFT

- Review
- DFT → DTFT
- k vs ω vs f
  DFT   DTFT   CT

Review

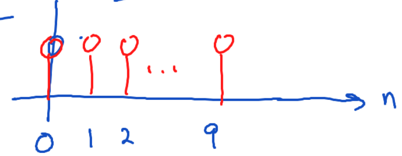
given $x[n]$ length N $n = 0, 1, 2, \dots N-1$

DFT $X[k]$ is N samples of $X(e^{j\omega})$ between 0 and "just under" 2π
exactly $\frac{N-1}{N} 2\pi$

ie sample $\omega = \frac{2\pi}{N} k$, $k = 0, 1, 2, \dots N-1$

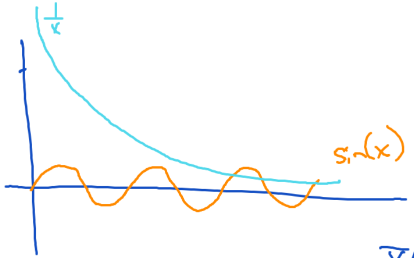
small x, $\sin(x) \approx x$

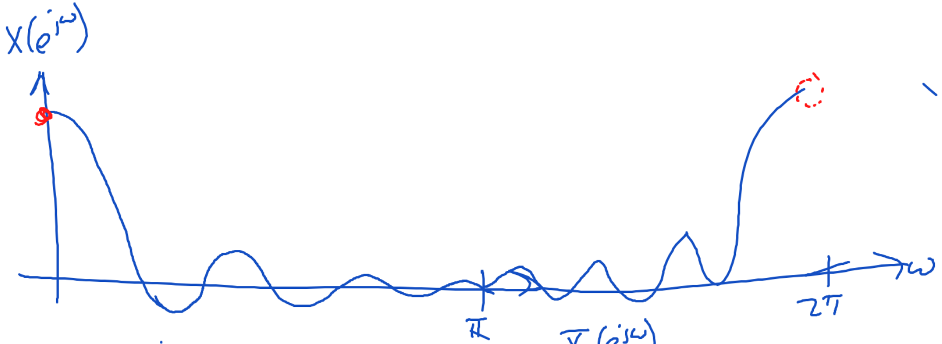
Ex $x[n]$ $N = 10$



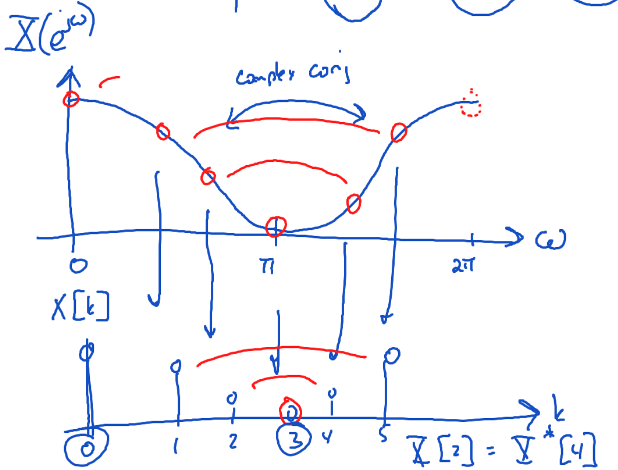
⟺ "Sinc" $\frac{k_1 \sin(k_2 x)}{x}$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{x}{x} = 1$$

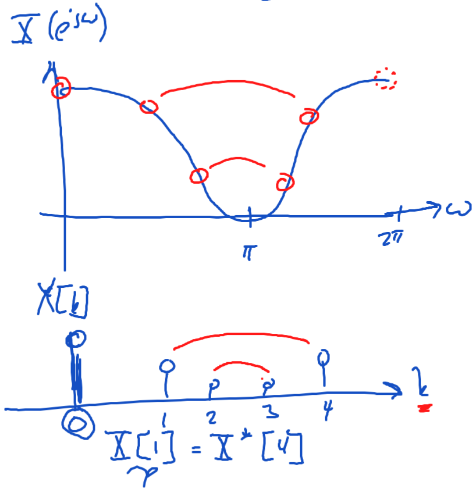




$N = 6$



$N = 5$



Zero padding

Adding zeros to end of x[n] so DFT more finely samples the DTFT.

Ex >> x already defined
>> xpad = [x zeros(1,100)]
>> X = fft(xpad)

MATLAB syntax

fft(x, N)

- Not same as N zeros at end!
- whole sequence of x & 0's are length N

Why
- more finely sample DTFT
- fft faster when N is power of 2

$N = 2^{24} - 1$
$N = 2^{24}$

xp = [zeros(1, 100) x]

$x[n] \longleftrightarrow X(e^{j\omega})$

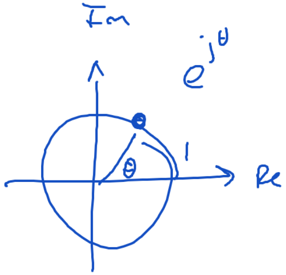
$xp[n] = x[n-100] \longleftrightarrow X(e^{j\omega})e^{j100\omega}$

$$|X(e^{j\omega})e^{j100\omega}| = |X(e^{j\omega})|\,\overbrace{|e^{j100\omega}|}^{}$$

$1 \angle 100\omega^\circ$

$= |X(e^{j\omega})|$

$\angle X(e^{j\omega})e^{j100\omega} = \angle X(e^{j\omega}) + 100\omega$



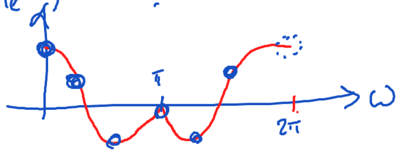
DFT → DTFT

$X[k] \rightarrow X(e^{j\omega})$

N all ω

DTFT $\xrightarrow{\text{sample}}$ DFT $\omega = \frac{2\pi}{N} k$, $k = 0, 1, 2, \ldots N-1$

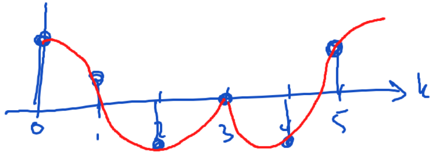
$X(e^{j\omega})$



$N = 6$

DFT ⟶ DTFT

$X[k]$



we know $x[n] \xrightarrow[\text{math}]{\text{DTFT}} X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$

we know $X[k] \xrightarrow[\text{math}]{\text{IDFT}} x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k n}{N}}$

goal $\underbrace{\text{DTFT}\left(\underbrace{\text{IDFT}\left(X[k]\right)}_{x[n]}\right)}_{X(e^{j\omega})}$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k n}{N}} \right) e^{-j\omega n}$$

DFT DTFT $x(t)$

k ω f

$$\boxed{\frac{f}{f_s} = \frac{\omega}{2\pi} = \frac{k}{N}}$$

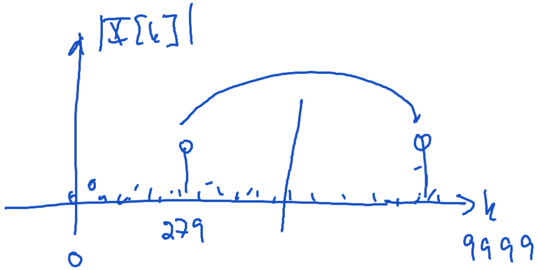
Ex

$f_s = 1$ k samples/sec $= \underline{1\text{ kHz}}$

$N = 10{,}000$

$x[n]$

stem(abs(fft))



$$\frac{f}{1000} = \frac{279}{10000}$$

$$\underline{f = 27.9\text{ Hz}}$$

Summary

To find DTFT $X(e^{j\omega})$

- intuitively: look at sequence

- analytically: $\sum_{n=0}^{N-1} x[n]e^{-j\omega}$
- computationally: fft(x, 2^30)

  $0 \sim 2\pi\frac{N-1}{N}$
- from DFT: $\sum\sum$ rarely used

To find DFT $X[k]$

- analytically $\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}}$
- computationally fft(x)
- from DTFT $X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N},\ k=0,1,\dots N-1}$